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Solving cubic and fourth degree equations.

1 Solving a cubic equation.

1.1 From a general equation to a simplified one.

In general, the cubic equation is defined in the form.

$$ax^3 + bx^2 + cx + d = 0, \text{where } a \neq 0 \tag{1}$$

Let us assume that the numbers a, b, c, d are complex. By making the substitution $y = x + b/(3a)$ the equation (1) will be simplified to the following form

$$y^3 + py + q = 0 \tag{2}$$

where $p = c/a - b^2/(3a^2)$ and $q = 2b^3/(3a)^3 - bc/(3a^2) + d/a$.

1.2 Solution of a simplified equation.

We will look for the solution of the equation (2) as a sum of two components $y = u + v$, then $y^3 = u^3 + v^3 + 3uv(u + v)$, hence $y^3 + py + q = u^3 + v^3 + (3uv + p)(u + v) + q = 0$. If both conditions are satisfied

$$\begin{cases} u^3 + v^3 = -q \\ uv = -p/3 \end{cases} \tag{3}$$

then $y = u + v$ will be the solution of the equation (2). Suppose that one of the solutions u or v is 0. It follows from the second equation of the system that this is possible only when $p = 0$, then the equation (2) has the form $y^3 + q = 0$. Its solutions will be $y = \sqrt[3]{-q}$. There will be three distinct roots in the complex numbers if $q \neq 0$ and three congruent zero roots when $q = 0$.

Now consider the case when $p \neq 0$, in this case there are no zero solutions of the system (3), so we can express v from the second equation and substitute it into the first, we get the equation $u^3 - p^3/(27u^3) + q = 0$. Doing variable substitution $t = u^3$ we get the quadratic equation $t^2 + qt - p^3/27 = 0$. Solving this quadratic equation we obtain t and find $u = \sqrt[3]{t}$.

$$u = \sqrt[3]{t} = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2}} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Since $v = \sqrt[3]{-q - u^3}$ we get

$$y = u + v = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

If we take one of the values for the root of $\sqrt{q^2/4 + p^3/27}$ in the formula for u, then for v it must be taken with a different sign to satisfy the condition $u^3 + v^3 = -q$. Changing the sign causes permutation of components, so any value can be taken as the root of $\sqrt{q^2/4 + p^3/27}$ and the formula becomes as follows.

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Since $uv = -p/3$, the second summand $v = -p/(3u)$ is uniquely defined.

1.3 The general algorithm.

We simplify the general equation $ax^3 + bx^2 + cx + d = 0$ to the reduced form $y^3 + py + q = 0$ where $p = c/a - b^2/(3a^2)$ and $q = 2b^3/(3a)^3 - bc/(3a^2) + d/a$.

If $p = 0$, then the solutions of the equation are $y_i = \sqrt[3]{-q}$.

If $p \neq 0$, then take any value of the complex root $d = \sqrt{q^2/4 + p^3/27}$, then calculate $u_i = \sqrt[3]{-q/2 + d}$, since $p \neq 0$, there will always be three complex roots. The roots will be $y_i = u_i - p/(3u_i)$.

From y_i we find the roots of the original equation $x_i = y_i - b/(3a)$.

1.4 Extras.

Let us study the function $f(y) = y^3 + py + q$ where p and q are real numbers. To do this, we find its derivative $f'(y) = 3y^2 + p$. If $p = 0$, there will be three coincident zero roots when $q = 0$, and if $q \neq 0$, there will be one real root. If $p > 0$, then $f'(y) > 0$ and so there will be one real root. If $p < 0$, then there will be two extremum points $y_i = \pm\sqrt{-p/3}$. Let's find the values at these points $f(y_i) = y_i(y_i^2+p)+q = \pm\sqrt{-p/3}\cdot 2p/3+q = 2(\pm\sqrt{-p^3/27}+q/2)$. Let's find $f(y_1)f(y_2) = 4(q^2/4+p^3/27) = 4D$. On the interval $(-\sqrt{-p/3}, \sqrt{-p/3})$ the derivative is less than zero, the function f is decreasing there, so if $f(y_1)$ and $f(y_2)$ have the same signs, i.e. $f(y_1)f(y_2) > 0$, the equation will have only one real root. If $f(y_1)$ and $f(y_2)$ have different signs, that is, $f(y_1)f(y_2) < 0$, then the equation have three real roots. Let's write out all cases in the table.

1. $f(y_1) = f(y_2) = 0$ three coincident real roots $D = 0$.
2. $f(y_1) = 0, f(y_2) \neq 0$ or $f(y_1) \neq 0, f(y_2) = 0$ two coincident real roots $D = 0$.
3. $f(y_1)f(y_2) > 0$ one real root $D > 0$.
4. $f(y_1)f(y_2) < 0$ three different real roots $D < 0$.

This table fits if $p \leq 0$. Above we have considered the case when $p > 0$, the table also fits for it, so the table works $\forall p$. Thus we get.

1. $D = 0, p = q = 0$ three coincident zero roots.
2. $D = 0, p \neq 0$ or $q \neq 0$ three real roots two of which coincide.
3. $D > 0$ one real and two complex roots.
4. $D < 0$ three different real roots.

1.5 Trigonometric solution.

1.5.1 Solution via sine.

Using the equality $\sin^3 a = 3\sin a/4 - \sin(3a)/4$, assume $y = r\sin a$ and substitute y into the equation $y^3 + py + q = 0$, we get

$$r^3\sin^3 a + pr\sin a + q = \frac{3r^3\sin a}{4} - \frac{r^3\sin(3a)}{4} + pr\sin a + q = 0$$

grouping the summands we obtain

$$-\frac{r^3\sin(3a)}{4} + \left(\frac{3r^3}{4} + pr\right)\sin a + q = 0$$

For the coefficient at $\sin a$ to be zero, the condition must be satisfied $3r^2 + 4p = 0$. Hence we obtain that $r = 2\sqrt{-p/3}$. Then $\sin(3a) = 4q/r^3 = q/(2\sqrt{-p^3/27}) = \frac{q}{2}\sqrt{\frac{27}{-p^3}}$. The equation will have three real roots if $p < 0$ and

$$\left|\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right| \le 1 \quad \Leftrightarrow \quad \frac{q^2}{4} \le \frac{-p^3}{27} \quad \Leftrightarrow \quad \frac{q^2}{4} + \frac{p^3}{27} \le 0$$

From here $3a = \arcsin\left(\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + 2\pi k$

$$a = \frac{1}{3}\left(\arcsin\left(\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + 2\pi k\right), \text{ where } k = 0, 1, 2$$

We obtain the solution of the equation

$$y = r\sin a = 2\sqrt{\frac{-p}{3}}\sin\left(\frac{1}{3}\arcsin\left(\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + \frac{2\pi k}{3}\right), \ k = 0, 1, 2$$

1.5.2 Solution via cosine.

$\cos(3a) = 4\cos^3 a - 3\cos a$. Hence $\cos^3 a = \cos(3a)/4 + 3\cos a/4$, assuming $y = r\cos a$, substituting y into the equation $y^3+py+q = 0$, we get

$$r^3\cos^3 a + pr\cos a + q = \frac{r^3\cos(3a)}{4} + \frac{3r^3\cos a}{4} + pr\cos a + q = 0$$

Let's make the coefficient in $\cos(a)$ equal to zero. $3r^2+4p=0$. Find the value of $r = 2\sqrt{-p/3}$. Then $\cos(3a) = -4q/r^3 = -\frac{q}{2}\sqrt{\frac{27}{-p^3}}$. Hence we obtain $3a = \arccos\left(-\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + 2\pi k$.

$$a = \frac{1}{3}\left(\arccos\left(-\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + 2\pi k\right), \text{ where } k = 0, 1, 2$$

We obtain the solution of the equation

$$y = r\cos a = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\sqrt{\frac{27}{-p^3}}\right) + \frac{2\pi k}{3}\right), \ k = 0, 1, 2$$

1.5.3 Example.

Consider the equation $x^3 - 7x - 6 = (x+1)(x+2)(x-3) = 0$. Solve it in trigonometric form $p = -7, \ q = -6$.

$$y = 2\sqrt{\frac{7}{3}}\sin\left(\frac{1}{3}\arcsin\left(-\frac{9}{7}\sqrt{\frac{3}{7}}\right) + \frac{2\pi k}{3}\right), \text{ where } k = 0, 1, 2$$

$$y = 2\sqrt{\frac{7}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{9}{7}\sqrt{\frac{3}{7}}\right) + \frac{2\pi k}{3}\right), \text{ where } k = 0, 1, 2$$

2 Solving an equation of the fourth degree.

2.1 From general equation to simplified equation.

In general, a fourth degree equation is defined in the form.

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \text{where } a \neq 0 \tag{4}$$

Substituting $y = x - b/(4a)$ we obtain the reduced equation

$$y^4 + py^2 + qy + r = 0 \tag{5}$$

with coefficients,

$$p = \frac{8ac - 3b^2}{8a^2}, \quad q = \frac{8a^2d - 4abc + b^3}{8a^3}$$

$$r = \frac{256a^3e - 64a^2bd + 16ab^2c - 3b^4}{256a^4}$$

2.2 Solving a simplified equation.

We add and subtract the summand $2sy^2 + s^2$, where s is some unknown variable.

$$(y^4+2sy^2+s^2)+py^2-(2sy^2+s^2)+qy+r = (y^2+s)^2+(p-2s)y^2+qy+r-s^2$$

For now we assume that $s \neq p/2$, then

$$(y^2+s)^2+(p-2s)y^2+qy+r-s^2 = (y^2+s)^2+(p-2s)\left(y^2 + \frac{qy}{p-2s}\right)+r-s^2$$

To make the expression in brackets a complete square, we need to add and subtract $q^2/(4(p-2s)^2)$.

$$(y^2+s)^2+(p-2s)\left(y^2 + \frac{qy}{p-2s} + \frac{q^2}{4(p-2s)^2}\right)+r-s^2-\frac{q^2}{4(p-2s)} = 0$$

Now we want to find s such that the free term is equal to zero $r - s^2 - q^2/(4(p-2s)) = 0$ or $(r - s^2)(p-2s) - q^2/4 = 0$. Thus we obtain a cubic equation for s.

$$2s^3 - ps^2 - 2rs + rp - q^2/4 = 0 \tag{6}$$

In this case, the simplified equation (5) reduces to the form

$$(y^2 + s)^2 + (p - 2s)\left(y + \frac{q}{2(p - 2s)}\right)^2 = 0$$

hence we obtain two quadratic equations

$$y^2 + s \pm \sqrt{p - 2s}\left(y + \frac{q}{2(p - 2s)}\right) = 0$$

Since we consider that $s \neq p/2$, the root of $\sqrt{p - 2s}$ has two values, that is, we get four solutions of two quadratic equations. Let's go back to the cubic equation $2s^3 - ps^2 - 2rs + rp - q^2/4 = 0$ (6). If it has at least one root s not equal to $p/2$, we can use it.

Now consider the case when all roots of the equation (6) are equal to $p/2$, that is, it has the form $2(s-p/2)^3 = 2s^3-3ps^2+\ldots = 0$, then all coefficients of this polynomial must be equal to the coefficients of the polynomial (6). In particular, the coefficient at s^2 must be the same. In one polynomial it is equal to $-p$ in the other $-3p$, that is, $p = 0$. Therefore, the polynomial has the form $2(s - p/2)^3 = 2s^3$ and to match the coefficients at all powers of s the values of q and r must be equal to zero. That is, in this case the above equation has the form $y^4 = 0$ and has four coincident zero roots.

2.3 The general algorithm.

We reduce the general equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ to the simplified equation $y^4+py^2+qy+r = 0$, where $p = (8ac-3b^2)/(8a^2)$, $q = (8a^2d - 4abc + b^3)/(8a^3)$, $r = (256a^3e - 64a^2bd + 16ab^2c - 3b^4)/(256a^4)$.

If $p = q = r = 0$, there will be four coinciding zero roots $y_i = 0$.

If at least one of the values of p, q, r is not equal to 0, then solve the cubic equation $2s^3 - ps^2 - 2rs + rp - q^2/4 = 0$. We choose a root s not equal to $p/2$, such a root is always present. Substitute s into the quadratic equations and find the four roots y_i.

$$y^2 + s \pm \sqrt{p - 2s}\left(y + \frac{q}{2(p - 2s)}\right) = 0$$

From y_i we find the roots of the original equation $x_i = y_i - b/(4a)$.